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Directed percolation, fractal roots and the Lee–Yang theorem

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Abstract

In the directed percolation model we consider the probability p of having an open bond as a complex parameter. We show that the roots of the survival probability $P_N(p)$ for a square lattice of N rows distribute themselves in a fractal manner in the complex p-plane. These roots have an accumulation point on the real axis which coincides with the critical probability $p_c = 0.6447$.

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Spreading phenomena abound in nature in such diverse situations as forest fires, epidemics or transport in random media (For a review on DP and recent literature see Ref. [1]). All these processes are characterized by a competition between particle reproduction and decay, controlled by some external parameter p. For instance, in the case of the spreading of a disease, the active phase is characterized by a stationary state in which infection and recovery balance one another. The simplest paradigm for such systems is directed percolation (DP) [2]. In this model, sites of a lattice can be either infected (active) or healthy (inactive) and, depending on the value of p, activity may either spread over the entire lattice or vanish after some time. In the case where inactivity takes over the system becomes trapped in an absorbing state that cannot be left. Thus, the process is out of equilibrium.

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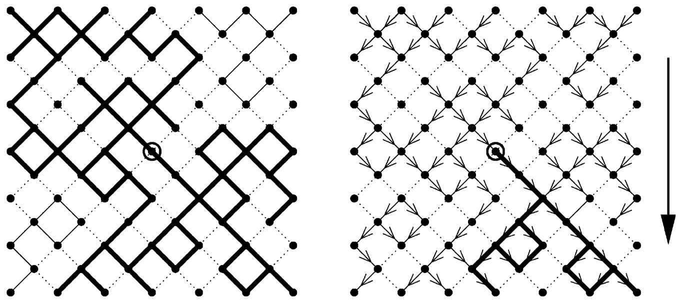

Fig. 1. Isotropic and directed percolation on a square lattice with free boundary conditions. A cluster (bold lines) is generated from the lattice site at the center (apex of a pyramid–shaped sublattice in DP, where spreading must follow the sense of the arrows).

Consider a square lattice as in the r.h.s. of Fig. 1. Let the bonds between edges have a probability p of being open and $1 - p$ otherwise. Two sites are connected if and only if, starting from an upper site, one may reach a lower site walking always in the allowed directions over a cluster. For the infinite lattice, there is a critical value of $p = p_c$ above which there will always be an infinite cluster, i.e., one which goes from the starting point all the way down the lattice. There will consequently be a non-zero probability $P(p)$ that some site O is connected to an infinite cluster through a path that runs downward from O.

The finite lattice with N rows and apex O has one site in the first row, two in the second and so on, up to a total of $N(N+1)/2$ sites. We define the survival probability $P_N(p)$ as the probability that at least one site in the bottom row is connected to the starting point (apex). For convenience, we represent the state of each site by $\sigma = 0$ (inactive) and $\sigma = 1$ (active). We define $W(\sigma_j|\sigma_k\sigma_l)$ as the probability that a given site j is in the state σ_j given that sites k and l immediately above j are in the states σ_k and σ_l, respectively. Considering the possible arrangements of bonds (k, j) and (l, j) connecting these three sites one may write down a general expression for $W(\sigma_j|\sigma_k\sigma_l)$

$$W(1|a,b) = 1 - (1-p)^{(a+b)} \quad W(0|a,b) = 1 - W(1|a,b)\,. \tag{1}$$

The survival probability $P_N(p)$ is given by

$$P_N(p) = \sum_{\text{all states}} \left(\prod_{\text{all plaquettes ijk}} W(\sigma_i|\sigma_j\sigma_k) \right) \prod_i \sigma_i\,, \tag{2}$$

where the last product runs over all spins i of the last row. At the edges of the pyramid–shaped lattice thus generated, plaquettes are connected to empty sites (free–bounday conditions). Given these definitions one may compute $P_N(p)$ successively for increasing values of N. The first few are given by the expressions below

$$P_1(p) = 1\,,$$
$$P_2(p) = -p^2 + p\,,$$

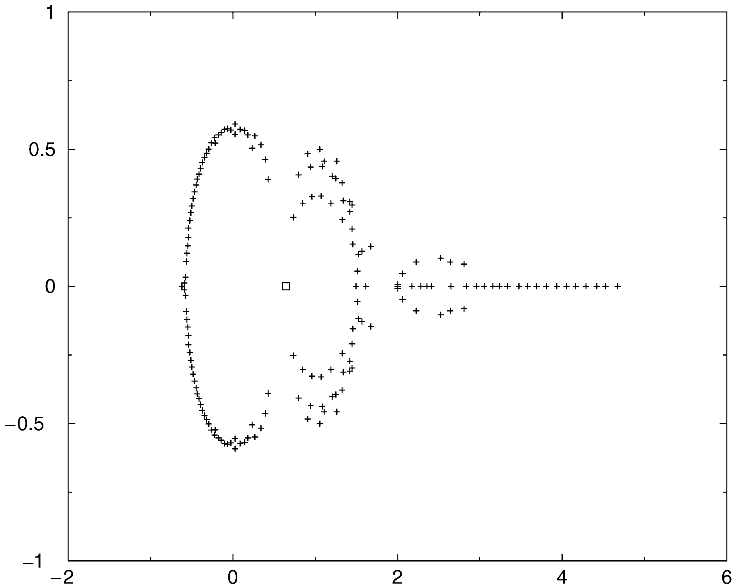

Fig. 2. Distribution of roots of the survival probability $P_N(p)$ for a square lattice with 13 rows (91 sites) in the complex p–plane. The roots are depicted as $+$ while the critical probability p_c is represented by a square ($\square$).

$$P_3(p) = -p^6 + 4p^5 - 4p^4 - 2p^3 + 4p^2 \,,$$
$$P_4(p) = p^{12} - 8p^{11} + 24p^{10} - 30p^9 + 5p^8 + 18p^7 - 3p^6 - 10p^5 - 4p^4 + 8p^3 \,. \tag{3}$$

Reinterpreting this model in terms of an Ising model with two– and three–spin interactions [3], where the survival probability is the partition function, a study of the zeros of $P_N(p)$, in the limit $N \to \infty$, should give information on the phase structure of the system according to the Lee-Yang theorem. Our numerical results for the roots of polynomials for a sequence of growing lattice sizes are depicted in Fig. 2, where we plot the distribution of roots in the complex p–plane for $N = 13$. One notices the two major curvilinear portions converging to a point on the real axis (represented by a square) within the physical region $0 < p < 1$. This convergence can be also seen in Fig. 3, where we plot the imaginary part of the root $p_0^{(N)}$ closest to the convergence point, as a function of the lattice size. By performing a standard numerical extrapolation on the sequence of values of $\Re(p_0^{(N)})$ and $\Im(p_0^{(N)})$ for $N = 3, 4, \ldots, 17$ we obtain the limiting values $\Re(p_0^{(\infty)}) = 0.64475 \pm 0.0005 = p_c$ and $\Im(p_0^{(\infty)}) = 0$.

Thus treating p as a complex variable and studying the properties of the zeros of $P_N(p)$ one gains essential information on the phase structure of the model. A more thorough study on the fractal distribution of zeros will of be published elsewhere [4].

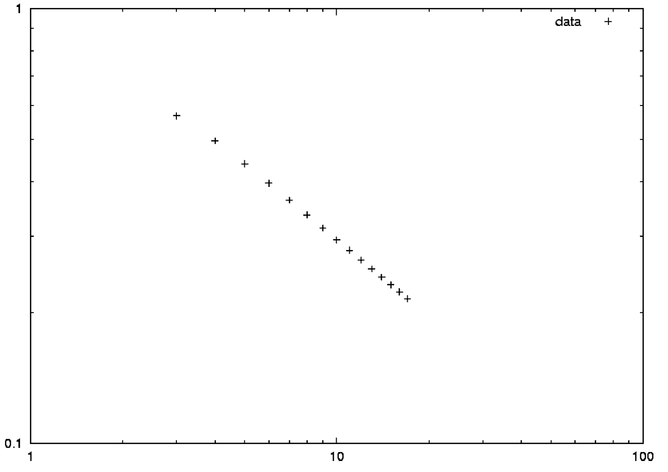

Fig. 3. Imaginary part of the root closest to the real axis in the inner portion of Fig. 2 as a function of N.

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